STABILITY OF AN AUTOMATICALLY CONTROLLED BICYCLE MOVING ON A HORIZONTAL PLANE

(USTOICHIVOST' AVTOMATICHESKI UPRAVLIAEMOGO VELOSIPEDA, KATIASHCHEGOSIA PO GORIZONTAL'NOI PLOSKOSTI)

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There exists a considerable amount of literature dealing with the motion of a bicycle on a horizontal plane.

At the present time this problem is of interest because of the possibility of utilizing the bicycle mechanism in systems containing automatically controlled bicycle chassis.

A method of investigating this problem, when the nonlinear law of the front wheel servo control is taken into account, is given below. Stability may be attained for any, even arbitrarily small, bicycle velocity provided that the velocity of the steering mechanism servomotor is sufficiently great.*

1. Statement of the problem. Let θ be the angle between the bicycle frame and the vertical, ψ be the angle between the front wheel and the line M_1M_2 connecting the points in which the bicycle wheels touch the horizontal plane.

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* During preparation of this article for publication, a reviewer noted that the stability investigation methods used for cases with large disturbances are applied here to a problem which has meaning only for small disturbances.

Although this is ture, the method presented here permits a generalization for the case of additional nonlinear elements, as well as for the case of stability of nonsteady motions, thus justifying its application to this particular problem.

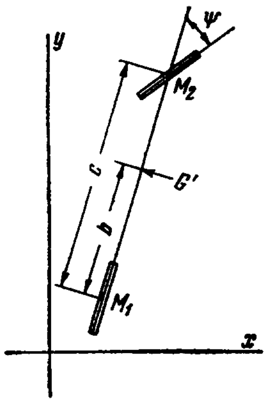

Fig. 1.

Let us assume that the wheels roll without slipping. Let us utilize the known (see, for example [1]) equation of the moment of momentum with respect to the M_1M_2 axis:

$$\ddot{\theta} = -\frac{bv}{cd}\dot{\phi} + \frac{g}{d}\left(\theta - \frac{v^2}{cg}\phi\right) \tag{1.1}$$

In the case of small deviations this equation describes a disturbed motion of the bicycle, observed in the vicinity of its steady-state motion with constant velocity v along the fixed straight line M_1M_2. Here d denotes the distance between the center of gravity G of the bicycle and its load and the line M_1M_2, c is the distance between the points M_1 and M_2, b is the distance between G' (G' is projection of the center of gravity G on the line M_1M_2) and the point M_1 where the rear wheel touches the horizontal plane (Fig. 1). Let us assume that the front wheel is steered automatically by a servomotor whose equation of motion is given by

$$\dot{\phi} = W\dot{\theta} + f(\sigma), \quad \sigma = a\theta + E\dot{\theta} + G^2\ddot{\theta} - \frac{1}{l}\phi - N\dot{\phi} \tag{1.2}$$

Here the constant W is characteristic of the gyroscopic moment of the front wheel, the stabilizing effect of which is well known [2] for the case of a freely rolling bicycle, a, E, G^2, l, and N are the control system parameters. The nonlinear function $f(\sigma)$ belongs either to the class A or class A′ of the functions of [3] (Fig. 2).

It is required to establish stability conditions for the undisturbed motion of a bicycle, which is controlled by a servomotor with the nonlinear characteristic $f(\sigma)$.

2. Stability when the rolling velocity of the bicycle is sufficiently large. Let us introduce the following notation for constants

$$\frac{g}{d} = m, \qquad n = \frac{v^2}{cd}, \qquad p = \frac{bv}{cd} \tag{2.1}$$

Obviously, we have

$$\ddot{\theta} = m\theta - pW\dot{\theta} - n\psi - pf(\sigma) \qquad \dot{\psi} = W\dot{\theta} + f(\sigma)$$

$$\sigma = (a + mG^2)\theta + [E - W(N + pG^2)]\dot{\theta} - (1/l + nG^2)\psi - (N + pG^2)f(\sigma) \tag{2.2}$$

Let us first reduce these equations to the normal Cauchy form. By means of the following notations

$$\theta = \eta_1, \qquad \frac{d\theta}{dt} = \sqrt{s}\,\eta_2, \qquad \psi = \eta_3, \qquad \tau = \sqrt{s}t \tag{2.3}$$

where the new variables are dimensionless and the constant s for the time being is left undefined, we obtain

$$\dot{\eta}_1 = \eta_2, \quad \dot{\eta}_2 = b_{21}\eta_1 + b_{22}\eta_2 + b_{23}\eta_3 + h_2 f(\sigma), \quad \dot{\eta}_3 = b_{32}\eta_2 + f(\sigma)$$

$$\sigma = p_1\eta_1 + p_2\eta_2 + p_3\eta_3 - (N + pG^2)f(\sigma) \tag{2.4}$$

The general stability problem described by equations of the form (2.4) has been treated elsewhere [4]. Here we use

$$\frac{m}{s} = b_{21}, \qquad -\frac{pW}{\sqrt{s}} = b_{22}, \qquad -\frac{M}{s} = b_{23}, \qquad -\frac{p}{s} = h_2$$

$$W = b_{22}, \qquad \frac{1}{\sqrt{s}} = h_3, \qquad p_1 = a + mG^2 \tag{2.5}$$

$$p_2 = [E - W(N + pG^2)]\sqrt{s}, \qquad p_3 = -\left(\frac{1}{l} + nG^2\right)$$

The expression for σ deserves special attention. When the moment with the constant p is taken into account in the bicycle equation, steering the bicycle in the direction of θ (acceleration of the fall) amplifies the effect of control achieved by tachometer feedback.

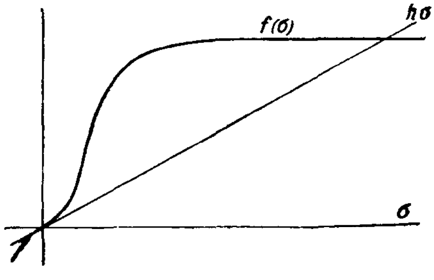

Fig. 2.

Let us reduce equations (2.4) to the canonic form. Let λ_1, λ_2, λ_3 be the roots of the equation

$$\lambda\,[\lambda^2 - b_{22}\lambda - b_{21} - b_{22}b_{23}] = 0$$

The canonic transformation shall be defined by the formulas

$$\begin{aligned}
x_1 &= \frac{b_{21}}{H(\lambda_1)}\eta_1 + \frac{\lambda_1}{H(\lambda_1)}\eta_2 + \frac{b_{23}}{H(\lambda_1)}\eta_3 \\
x_2 &= \frac{b_{21}}{H(\lambda_2)}\eta_1 + \frac{\lambda_2}{H(\lambda_2)}\eta_2 + \frac{b_{23}}{H(\lambda_2)}\eta_3 \\
x_3 &= -\frac{b_{32}}{h_3}\eta_1 + \frac{1}{h_3}\eta_3 \\
H(\lambda) &= h_2\lambda + b_{23}h_3 = h_2\lambda + b_{23}h_3
\end{aligned} \tag{2.6}$$

This transformation will not be a special one if $\lambda_1 \neq \lambda_2$ and thus, it can be carried out. The canonic equations in terms of the new variables shall assume the form:

$$\dot{x}_1 = \lambda_1 x_1 + f(\sigma), \quad \dot{x}_2 = \lambda_2 x_2 + f(\sigma), \quad \varkappa\dot{\sigma} = \beta_1 x_1 + \beta_2 x_2 - Rf(\sigma) \tag{2.7}$$

Here

$$\varkappa(\sigma) = 1 + (N + pG^2)\frac{df}{d\sigma}, \qquad \beta_1 = -\frac{H(\lambda_1)}{\lambda_2 - \lambda}[p_1 + \lambda_1 p_2 + b_{32}p_3]$$

$$R = -h_2 p_2 - h_3 p_3, \qquad \beta_2 = \frac{H(\lambda_2)}{\lambda_2 - \lambda_1}[p_1 + \lambda_2 p_2 + b_{32}p_3] \tag{2.8}$$

The numbers λ_1 and λ_2 are defined by the equalities

$$\lambda_1 + \lambda_2 = b_{22}, \qquad \lambda_1\lambda_2 = \frac{nW - m}{s} \tag{2.9}$$

Let us specify the condition

$$nW - m > 0 \tag{2.10}$$

Equation $x_3 = f(\sigma)$ for the canonic variable x_3 drops out of the system (2.7). The nonlinear function $\kappa(\sigma)$ is positive and $\mathrm{Re}(\lambda_1, \lambda_2) < 0$.

It is easy to see that stability with respect to the variables x_1, x_2, σ also guarantees stability with respect to the variables η_1, η_2, η_3. The problem is to choose the control system constants a, E, G^2, l, N such that stability for any $f(\sigma)$ of class A will be guaranteed.

Let us examine definite and everywhere positive function

$$V = -\frac{a_1^2}{2\lambda_1}x_1^2 - \frac{2a_1a_2}{\lambda_1 + \lambda_2}x_1x_2 - \frac{a_2^2}{2\lambda_2}x_2^2 + \int_0^\sigma \varkappa(\sigma) f(\sigma)\, d\sigma \tag{2.11}$$

Its total derivative in accordance with (2.7) is of the form:

$$\dot{V} = -(a_1x_1 + a_2x_2 + \sqrt{R}f(\sigma))^2$$

if the following relationships are satisfied

$$\beta_1 + 2\sqrt{R}a_1 - \frac{2a_1a_2}{\lambda_1+\lambda_2} - \frac{a_1^2}{\lambda_1} = 0, \quad \beta_2 + 2\sqrt{R}a_2 - \frac{2a_1a_2}{\lambda_1+\lambda_2} - \frac{a_2^2}{\lambda_2} = 0 \quad (2.12)$$

The equations (2.12) can be solved for a_1 and a_2, if the following inequalities are satisfied

$$\Gamma^2 = R + \frac{\beta_1}{\lambda_1} + \frac{\beta_2}{\lambda_2} > 0, \qquad R > 0 \qquad (2.13)$$

$$D^2 = (\lambda_1^2 + \lambda_2^2) R + \lambda_1\beta_1 + \lambda_2\beta_2 \pm 2\lambda_1\lambda_2 \sqrt{R\Gamma^2} > 0 \qquad (2.14)$$

Consequently, the stability criterion is reduced to the limiting of the choice of the control system parameters by the inequalities (2.13), (2.14), and also by the inequality (2.10). Let us examine them. The inequality $R > 0$ (2.13) has the form:

$$pE + 1/l + nG^2 > pW(N + pG^2) \qquad (2.15)$$

Furthermore,

$$\frac{\beta_1}{\lambda_1} + \frac{\beta_2}{\lambda_2} = h_2p_2 - \frac{h_3b_{23}}{\lambda_1\lambda_2}(p_1 + b_{32}p_3)$$

$$\lambda_1\beta_1 + \lambda_2\beta_2 = (h_2b_{22} + h_2b_{23})\,p_1 + [h_2(b_{22}^2 + b_{21} + b_{23}b_{32}) + h_3b_{22}b_{23}]\,p_2 + (h_2b_{22} + h_3b_{23})\,b_{32}p_3$$

Consequently,

$$\Gamma^2 = \frac{h_3(a\ln - m)}{l(Wn - m)}$$

From this we must have

$$aln - m > 0 \qquad (2.16)$$

The reversal of the inequality is not possible by virtue of (2.10). Finally, using (2.8), (2.5), and (2.1), we determine the last stability condition:

$$D^2 = (p^2W - n)\,a + pnW[E - W(N + pG^2)] + m\left(\frac{1}{l} + np^2G^4\right) \pm 2\left\{(Wn - m)\left(\frac{aln - m}{l}\right)\left[\frac{1}{l} + nG^2 + pE - pW(N + pG^2)\right]\right\}^{1/2} > 0 \qquad (2.17)$$

The derived conditions can be interpreted mechanically.

If the wheel parameters are given, the constant for the gyroscopic moment is $W = kv$, where k is a proportionality constant, and the inequality (2.10) assumes the form:

$$kv^3 > cg \qquad (2.18)$$

This inequality will be satisfied if the bicycle velocity is sufficiently large. Let us note that the inequality (2.18) constitutes the condition of the bicycle self-stabilization due to the gyroscopic moment of the wheel of a freely rolling bicycle [2]. It imposes a lower limit upon the bicycle velocity and it does not contain any control system parameters. In this case the inequality (2.16) imposes a lower limit upon the control system parameter al and we have

$$alV^2 > cg \tag{2.19}$$

The inequalities (2.15) and (2.17) can be satisfied by properly choosing E and G^2.

3. Stability for arbitrarily small bicycle velocity. Let us now suppose that the inequality (2.10) characterizing the presence of gyroscopic self-stabilization produced by the front wheel is not satisfied. Let

$$\gamma = N + pG^2 + \frac{h_2 p_3}{b_{23}}, \qquad b_{21}^\circ = b_{21} - \frac{p_1 b_{23}}{p_3}, \qquad b_{22}^\circ = b_{22} - \frac{p_2 b_{23}}{p_3} \tag{3.1}$$

After eliminating η_3 and using (2.4) we find

$$\dot{\eta}_1 = \eta_2, \qquad \dot{\eta}_2 = b_{21}^\circ \eta_1 + b_{22}^\circ \eta_2 + \frac{b_{23}}{p_3}[\sigma + \gamma f(\sigma)]$$

By means of a non-singular transformation these equations are reduced to the following

$$\dot{x}_s = \mu_s x_s + [\sigma + \gamma f(\sigma)] \qquad (s = 1, 2) \tag{3.2}$$

Here μ_s are the roots of the equation $\mu^2 - b_{22}{}^0\mu - b_{21}{}^0 = 0$. Since

$$-b_{22}^\circ = \frac{pW/l + n[E - WN]}{V s (1/l + nG^2)}, \qquad -b_{21}^\circ = \frac{aln - m}{ls(1/l + nG^2)} \tag{3.3}$$

then, specifying that the following inequalities are satisfied,

$$\frac{aln - m}{ls(1/l + nG^2)} > 0, \qquad \frac{pW/l + n[E - WN]}{1/l + nG^2} > 0 \tag{3.4}$$

we obtain

$$\operatorname{Re} \mu_k < 0 \qquad (k = 1, 2)$$

The desired transformation, after inversion, may be written as follows:

$$\eta_1 = \frac{b_{23}}{p_3(\mu_2 - \mu_1)}(x_2 - x_1), \qquad \eta_2 = \frac{b_{23}}{p_3(\mu_2 - \mu_1)}(\mu_2 x_2 - \mu_1 x_1) \tag{3.5}$$

The third equation is found by differentiating with respect to σ. Let

$$f(\sigma) = h\sigma + \varphi(\sigma) \tag{3.6}$$

where $\phi(\sigma)$ is a function of class A'

$$\left(\frac{df}{d\sigma}\right)_{\sigma=0} \geqslant h > 0 \tag{3.7}$$

Let us introduce the notations

$$
\begin{gathered}
1 + h\gamma = M, \qquad S = -\frac{b_{23}p_2}{p_3} - Rh, \quad R = -h_3p_3 - \frac{p_2b_{23}\gamma}{p_3} \\
\beta_1{}^\circ = \frac{b_{23}}{p_3(\mu_2 - \mu_1)}\left[-p_2b_{21}{}^\circ - \mu(p_1 + p_2b_{22}{}^\circ - p_3b_{32})\right] \\
\beta_2{}^\circ = \frac{b_{23}}{p_3(\mu_2 - \mu_1)}\left[p_2b_{21}{}^\circ + \mu_2(p_1 + p_2b_{22}{}^\circ + p_3b_{32})\right]
\end{gathered}
\tag{3.8}
$$

Then, finally, we get

$$
\begin{gathered}
\dot{x}_1 = \mu_1x_1 + [M\sigma + \gamma\varphi(\sigma)], \qquad \dot{x}_2 = \mu_2x_2 + [M\sigma + \gamma\varphi(\sigma)] \\
\varkappa\dot{\sigma} = \beta_1{}^\circ x_1 + \beta_2{}^\circ x_2 - S\sigma - R\varphi(\sigma)
\end{gathered}
\tag{3.9}
$$

Let us assume that

$$
M > 0, \qquad \gamma \geqslant 0, \qquad R > 0, \qquad S > 0 \tag{3.10}
$$

and examine the V-function

$$
V = -\frac{a_1{}^2x_1{}^2}{2\mu_1} - \frac{2a_1a_2}{\mu_1 + \mu_2}x_1x_2 - \frac{a_2{}^2}{2\mu_2}x_2{}^2 - \int_0^\sigma [M\sigma + \gamma\varphi(\sigma)]\,\varkappa(\sigma)\,d\sigma \tag{3.11}
$$

This function is positive everywhere except at the origin. Its total derivative computed in accordance with equations (3.9) has the form:

$$
\dot{V} = -(a_1x_1 + a_2x_2)^2 - [M\sigma + \gamma\varphi(\sigma)]\,[S\sigma + R\varphi(\sigma)]
$$

if the control system parameters are chosen such that the following equalities hold

$$
\beta_2{}^\circ - \frac{a_1{}^2}{\mu_1} - \frac{2a_1a_2}{\mu_1 + \mu_2} = 0, \qquad \beta_2{}^\circ - \frac{a_2{}^2}{\mu_2} - \frac{2a_1a_2}{\mu_1 + \mu_2} = 0 \tag{3.12}
$$

Together with (3.10) and (3.4) the stability conditions will be of the form:

$$
\Gamma^2 = \frac{\beta_1{}^\circ}{\mu_1} + \frac{\beta_2{}^\circ}{\mu_2} > 0, \qquad D^2 = \beta_1{}^\circ\mu_1 + \beta_2{}^\circ\mu_2 > 0 \tag{3.13}
$$

4. Stability conditions and their interpretation. First of all we have

$$
\gamma = N - \frac{b}{vl} \geqslant 0 \tag{4.1}
$$

This condition is easily satisfied by properly choosing a suitable value of the tachometer feedback constant N. In particular, in the absence of a rigid feedback link ($l = \infty$) this constant must be non-negative; in the rest of the discussion we take $N = \rho n$, where ρ is a positive constant.

If (4.1) is satisfied then, obviously, $M > 0$. Two last inequalities of (3.10) can be written as follows:

$$\left(\frac{1}{l} + nG^2\right)^2 > n\left(N - \frac{b}{vl}\right)[E - W(N + pG^2)], \quad h > \frac{n[E - W(N + pG^2)]}{RV\overline{s}\,(1/l + nG^2)} \tag{4.2}$$

If the bicycle control system has no feedback link with constant gain, the inequalities (3.4), (4.1), and (4.2) become obvious; the inequality (4.2) restricts the choice of the servomotor characteristic to the A′ class of functions.

Indeed, for the inequalities (4.2) we have

$$G^2 > \frac{\rho[E - W(N + pG^2)]}{G^2}, \qquad h > \frac{[E - W(N + pG)]\,G^2}{\{G^4 - \rho[E - W(N + pG^2)]\}\,n} \tag{4.3}$$

If the bicycle velocity is sufficiently small these inequalities can be simplified as follows:

$$G^2 > \frac{\rho E}{G^2}, \qquad h > \frac{G^2 Ecd}{(G^4 - \rho E)\,v^2} \tag{4.4}$$

The first inequality can be satisfied by properly choosing ρ; the second inequality imposes a further restriction upon the servomotors, namely, that in addition to having characteristics of the A′ class of functions, they must have velocity with the lower bound inversely proportional to the square of the bicycle velocity. Consequently, in order to retain stability at arbitrarily small bicycle velocity v, the servomotor must be able to rotate the front wheel arbitrarily fast. (The possibility of this solution was indicated by N.G. Chetaev).

Another solution can be obtained after making a simplifying assumption $G^2 = 0$. In this case the inequalities (3.13) will assume the form

$$E - W(N + pG^2) > 0 \tag{4.5}$$

$$-\left(an - \frac{m}{l}\right)[E - W(N + pG^2)] + \left[\frac{pW}{l} + n(E - WN)\right]\Big\{W(1/l + nG^2) + \\ + \frac{pW/l + n(E - WN)}{1/l + nG^2}[E - W(N + pG^2)] - (a + mG^2)\Big\} > 0 \tag{4.6}$$

In a special case ($l = \infty$) the inequality (4.6) can be simplified:

$$-a(E - WpG^2) + E\left\{nWG^2 + \frac{E}{G^2}(E - WpG^2) - a - mG^2\right\} > 0 \tag{4.7}$$

Clearly, conditions (4.6) and (4.7) will be satisfied for sufficiently large values of the artificial damping constant E.

The problem can be generalized to include the case of variable bicycle velocity. The methods for solution of the non-steady motions problem indicated in [3], Chapter XI and in [5], Chapter IX, may be applied in this case.

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